JOHN DISK AND K-QUASICONFORMAL HARMONIC MAPPINGS

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Abstract. The main aim of this article is to establish certain relationships between K-quasiconformal harmonic mappings and John disks. The results of this article are the generalizations of the corresponding results of Ch. Pommerenke [18].

1. Introduction and main results

For $a \in \mathbb{C}$ and $r > 0$, we let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ so that $\mathbb{D}_r := \mathbb{D}(0, r)$ and thus, $\mathbb{D} := \mathbb{D}_1$ denotes the open unit disk in the complex plane $\mathbb{C}$. This paper provides a necessary and sufficient condition for the image $\Omega = f(\mathbb{D})$ of univalent harmonic mappings f defined on $\mathbb{D}$ to be a *John disk* (see Theorems 1 and 2). Some differential properties of K-quasiconformal harmonic mappings will also be characterized by using *Pommerenke interior domains* and John disks (see Theorem 4 and Corollary 1). In addition, we present a sufficient condition, in terms of harmonic analog of the *pre-Schwarzian* of K-quasiconformal harmonic mappings f on $\mathbb{D}$, for $\Omega = f(\mathbb{D})$ to be a John disk (see Theorem 5). Similar results for analytic functions are proved earlier by Ahlfors and Weill [1], Becker and Pommerenke [2], and Pommerenke [18]. In order to state and prove our main results and related investigations, we need to recall some basic definitions, remarks and some results.

For a real 2×2 matrix A, we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued functions $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\overline{z}}| \ \text{ and } \ l(D_f) = \big||f_z| - |f_{\overline{z}}|\big|,$$

where $f_z = (1/2)\big(f_x - if_y\big)$ and $f_{\overline{z}} = (1/2)\big(f_x + if_y\big)$.

Let Ω be a domain in $\mathbb{C}$, with non-empty boundary. A sense-preserving homeomorphism f from a domain Ω onto Ω', contained in the Sobolev class $W^{1,2}_{loc}(\Omega)$, is

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said to be a K-*quasiconformal mapping* if, for $z \in \Omega$,

$$\|D_f(z)\|^2 \leq K \det D_f(z), \text{ i.e., } \|D_f(z)\| \leq Kl\big(D_f(z)\big),$$

where $K \geq 1$ and $\det D_f$ is the determinant of D_f (cf. [12, 14, 22, 23]).

A complex-valued function f defined in a simply connected subdomain G of $\mathbb{C}$ is called a *harmonic mapping* in G if and only if both the real and the imaginary parts of f are real harmonic in G. It is indeed a simple fact that every harmonic mapping f in G admits a decomposition $f = h + \overline{g}$, where h and g are analytic in G. If we choose the additive constant such that $g(0) = 0$, then the decomposition is unique. Since the Jacobian $\det D_f$ of f is given by

$$\det D_f := |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2,$$

f is locally univalent and sense-preserving in G if and only if $|g'(z)| < |h'(z)|$ in G; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in G (see [15] and also [8]).

In the recent years, the family $\mathcal{S}_H$ of all sense-preserving planar harmonic univalent mappings $f = h + \overline{g}$ in $\mathbb{D}$, with the normalization $h(0) = g(0) = 0$ and $h'(0) = 1$, attracted the attention of many function theorists. This class together with a few other geometric subclasses, originally investigated in details by [6], became instrumental in the study of univalent harmonic mappings. See the monograph [8] and the recent survey [20] for the theory of these functions.

If the co-analytic part g is identically zero in the decomposition of f, then the class $\mathcal{S}_H$ reduces to the classical family $\mathcal{S}$ of all normalized analytic univalent functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$. If $\mathcal{S}_H^0 = \{f = h + \overline{g} \in \mathcal{S}_H :\ g'(0) = 0\}$, then the family $\mathcal{S}_H^0$ is both normal and compact (see [6, 8, 20]).

Let $d_\Omega(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$ of Ω. In particular, we always use $d(z)$ to denote the Euclidean distance from z to the boundary $\partial\mathbb{D}$ of $\mathbb{D}$.

Definition 1. A bounded simply connected plane domain G is called a c-*John disk* for $c \geq 1$ with *John center* $w_0 \in G$ if for each $w_1 \in G$ there is a rectifiable arc γ, called a *John curve*, in G with end points w_1 and w_0 such that

$$\sigma_\ell(w) \leq c d_G(w)$$

for all w on γ, where $\gamma[w_1, w]$ is the subarc of γ between w_1 and w, and $\sigma_\ell(w)$ is the Euclidean length of $\gamma[w_1, w]$ (see [11, 9, 17, 19]).

Remark 1. If f is a complex-valued and univalent mapping in $\mathbb{D}$, $G = f(\mathbb{D})$ and, for $z \in \mathbb{D}$, $\gamma = f([0, z])$ in Definition 1, then we call c-John disk as a *radial* c-John disk, where $w_0 = f(0)$ and $w = f(z)$. In particular, if f is a conformal mapping, then we call c-John disk as a *hyperbolic* c-John disk. It is well known that any point $w_0 \in G$ can be chosen as John center by modifying the constant c if necessary. Moveover, if we don't emphasize the constant c, we regard the c-John disk as the John disk (cf. [11, 9, 17]).

In [18] (see also [19, p. 97]), Pommerenke proved that if f maps $\mathbb{D}$ conformally onto a bounded domain G, then G is a hyperbolic John disk if and only if there exist

constants $M > 0$ and $\delta \in (0,1)$ such that for each $\zeta \in \partial\mathbb{D}$, and for $0 \le r_1 \le r_2 < 1$, we have

$$|f'(r_2\zeta)| \le M|f'(r_1\zeta)| \left(\frac{1-r_2}{1-r_1}\right)^{\delta-1}.$$

Later, in [9, Theorem 2.3], Kari Hag and Per Hag gave an alternate proof of this result. In this paper, our first aim is to extend this result to planar harmonic mappings.

Theorem 1. *For $K \ge 1$, let $f \in \mathcal{S}_H^0$ be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto a bounded domain Ω. Then Ω is a radial John disk if and only if there are constants $M(K) > 0$ and $\delta \in (0,1)$ such that for each $\zeta \in \partial\mathbb{D}$ and for $0 \le r \le \rho < 1$,*

$$\|D_f(\rho\zeta)\| \le M(K)\|D_f(r\zeta)\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1}. \tag{1.1}$$

The following result is another characterization of radial John disk, which is also a generalization of [18, Theorem 1].

Theorem 2. *For $K \ge 1$, let $f \in \mathcal{S}_H^0$ be a K-quasiconformal mapping and $\Omega = f(\mathbb{D})$ is a bounded domain. Then the following conditions are equivalent:*

(a) *Ω is a radial John disk;*
(b) *There is a positive constant M_1 such that, for all $z \in \mathbb{D}$,*

$$\operatorname{diam} f(B(z)) \le M_1 d_\Omega(f(z)),$$

where $B(z) = \{\zeta : \ |z| \le |\zeta| < 1, \ |\arg z - \arg\zeta| \le \pi(1-|z|)\}$;
(c) *There is a positive constant $\delta \in (0,1)$ such that, for all $z \in \mathbb{D}$ and $\zeta \in B(z)$,*

$$\|D_f(\zeta)\| \le M_2\|D_f(z)\| \left(\frac{1-|\zeta|}{1-|z|}\right)^{\delta-1}, \tag{1.2}$$

where M_2 is a positive constant.

By using some distortion conditions in Theorem 2, we get a characterization of coefficients of K-quasiconformal harmonic mappings.

Theorem 3. *For $K \ge 1$, let $f = h + \overline{g} \in \mathcal{S}_H^0$ be a K-quasiconformal harmonic mapping, where*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=2}^{\infty} b_n z^n.$$

If f satisfies the condition (b) *or* (c) *in Theorem* 2*, then there is some $\beta_0 > 0$ such that*

$$\sum_{n=2}^{\infty} n^{1+\beta_0}(|a_n|^2 + |b_n|^2) < \infty.$$

Using Theorems 2 and 3, it can be easily seen that the conclusion of Theorem 3 continues to hold if the assumption that "f satisfies the condition (b) or (c) in Theorem 2" is replaced by "$\Omega = f(\mathbb{D})$ is a radial John disk".

For $p \in (0,\infty]$, the *generalized Hardy space* $H^p_g(\mathbb{D})$ consists of all those functions $f:\ \mathbb{D}\to\mathbb{C}$ such that f is measurable, $M_p(r,f)$ exists for all $r\in(0,1)$ and $\|f\|_p<\infty$, where

$$\|f\|_p=\begin{cases}\sup\limits_{0<r<1} M_p(r,f) & \text{if } p\in(0,\infty)\\ \sup\limits_{z\in\mathbb{D}}|f(z)| & \text{if } p=\infty\end{cases}, \quad \text{and} \quad M_p^p(r,f)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta.$$

Let $f\in\mathcal{S}_H$ be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto a domain G. For $0<r<1$ and $w_1,w_2\in f(\partial\mathbb{D}_r)$, let γ_r be the smaller subarc of $f(\partial\mathbb{D}_r)$ between w_1 and w_2, and let

$$d_{G_r}(w_1,w_2)=\inf_{\Gamma}\operatorname{diam}\Gamma,$$

where Γ runs through all arcs from w_1 to w_2 that lie in $G_r=f(\mathbb{D}_r)$ except for their endpoints. If

$$\sup_{0<r<1}\left\{\sup_{w_1,w_2\in\gamma_r}\frac{\ell\big(\gamma_r[w_1,w_2]\big)}{d_{G_r}(w_1,w_2)}\right\}<\infty, \tag{1.3}$$

then we call G as a *Pommerenke interior* domain (cf. [18]). In particular, if G is bounded, then we call G as a bounded Pommerenke interior domain. Our next theorem is an analogous result of [18, Theorem 3].

Theorem 4. *Let $f\in\mathcal{S}_H$ be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto a bounded Pommerenke interior domain G. If there are constants M and $\delta\in(0,1)$ such that for each $\zeta\in\partial\mathbb{D}$ and for $0\leq r\leq\rho<1$,*

$$\|D_f(\rho\zeta)\|\leq M\|D_f(r\zeta)\|\left(\frac{1-\rho}{1-r}\right)^{\delta-1}, \tag{1.4}$$

then $\|D_f\|\in H^1_g(\mathbb{D})$.

The following result easily follows from Theorems 1 and 4.

Corollary 1. *Let $f\in\mathcal{S}^0_H$ be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto a bounded Pommerenke interior domain G. If G is a radial John disk, then $\|D_f\|\in H^1_g(\mathbb{D})$.*

In terms of the canonical representation of a sense-preserving harmonic mappings $f=h+\overline{g}$ in $\mathbb{D}$ with $\omega=g'/h'$, as in the works of Hernández and Martín [10], the Pre-Schwarzian derivative P_f of f and the Schwarzian derivative S_f of f are defined by

$$P_f=T_h-\frac{\omega'\overline{\omega}}{1-|\omega|^2},\quad \text{and } S_f=Sh+\frac{\overline{\omega}}{1-|\omega|^2}\left(T_h\omega'-\omega''\right)-\frac{3}{2}\left(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\right)^2,$$

respectively. Here

$$T_h=\frac{h''}{h'}\quad\text{and}\quad Sh=T_h'-\frac{1}{2}T_h^2$$

are referred to as the Pre-Schwarzian and Schwarzian (derivatives) of a locally univalent analytic function f in $\mathbb{D}$, respectively. For the original definition of the Schwarzian derivative of harmonic mappings, see [4].

Ahlfors and Weill [1], Becker and Pommerenke [2] characterized the quasidisk by using the Pre-Schwarzian of analytic functions. On the basis of the works of Chuaqui, et al. [5], Kari Hag and Per Hag [9] discussed the relationships between the John disk and the Pre-Schwarzian of analytic functions. It is natural to ask whether a similar relationship is attainable (see [5, Theorem 4] and [9, Theorem 3.7]) with the help of Pre-Schwarzian of harmonic mappings. This is the content of our next result.

Theorem 5. *Suppose that $f \in \mathcal{S}_H^0$ is a K-quasiconformal harmonic mapping of $\mathbb{D}$ onto a bounded domain $f(\mathbb{D})$ for some $K \geq 1$ and such that*

$$\lim_{|z|\to 1^-} \sup \left\{(1-|z|^2)\mathrm{Re}\big(zP_f(z)\big)\right\} < 1.$$

If $\ell\big(f([0,z])\big) < \infty$ for all $z \in \mathbb{D}$, then $f(\mathbb{D})$ is a radial John disk.

The proofs of Theorems 1−5 will be given in Section 2.

2. The proofs of the main results

We begin the section by recalling the following results which play an important role in the proofs of Theorems 1−5.

Theorem A. ([13, Proposition 3.1] and [13, Theorem 3.2]) *Let f be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto itself. Then for all $z \in \mathbb{D}$, we have*

$$\frac{1+K}{2K}\left(\frac{1-|f(z)|^2}{1-|z|^2}\right) \leq |f_z(z)| \leq \frac{K+1}{2}\left(\frac{1-|f(z)|^2}{1-|z|^2}\right).$$

Theorem B. ([3, Theorem 3]) *Let $f \in \mathcal{S}_H^0$. Then there is a positive constant $c_1 < +\infty$ such that for $\xi \in \partial\mathbb{D}$ and $0 \leq r_3 \leq r_4 < 1$,*

$$\|D_f(r_4\xi)\| \geq \frac{1}{2^{1+c_1}}\|D_f(r_3\xi)\|\left(\frac{1-r_4}{1-r_3}\right)^{c_1-1}.$$

Proof of Theorem 1. We first prove the sufficiency. Applying [16, Proposition 13], we obtain that

$$\|D_f(z)\| \leq \frac{16Kd_\Omega(f(z))}{1-|z|^2}. \tag{2.1}$$

Also, by (1.1) and (2.1), for $w=f(r\zeta)$ and $w_1=f(\rho\zeta)$, we have

$$\begin{aligned}
\sigma_\ell(w) &= \int_r^\rho |df(t\zeta)| \leq \int_r^\rho \|D_f(t\zeta)\|dt\\
&\leq M(K)\|D_f(r\zeta)\|\int_r^1 \left(\frac{1-t}{1-r}\right)^{\delta-1} dt, \quad \text{by (1.1)},\\
&= \frac{M(K)}{\delta}\|D_f(r\zeta)\|(1-r)\\
&\leq \frac{M(K)}{\delta}\|D_f(r\zeta)\|(1-r^2)\\
&\leq \frac{16KM(K)}{\delta}d_\Omega(w), \quad \text{by (2.1)},
\end{aligned}$$

which implies that Ω is a radial $(16KM(K)/\delta)$-John disk with John center $w_0=f(0)$ and with $\gamma=f([0,\rho\zeta])$ as the John curves, where $r\in[0,1)$, $\rho\in[r,1)$ and $\zeta\in\partial\mathbb{D}$.

Now we prove the necessity. For $z\in\mathbb{D}$, let

$$\Delta = f^{-1}\Big(\mathbb{D}\big(f(z), d_\Omega(f(z))\big)\Big)$$

and ϕ be a conformal mapping of $\mathbb{D}$ onto Δ with $\phi(0)=z$. Since $\phi(\mathbb{D})\subset\mathbb{D}$, we know that, for $w\in\mathbb{D}$,

$$|\phi'(w)| \leq \frac{1-|\phi(w)|^2}{1-|w|^2}. \tag{2.2}$$

Then

$$F(w) = \frac{1}{d_\Omega(f(z))}\big(f(\phi(w))-f(z)\big)$$

is a K-quasiconformal harmonic mapping of $\mathbb{D}$ onto itself with $F(0)=0$. It is not difficult to know that

$$\|D_F(w)\| = \frac{|\phi'(w)|\|D_f(\phi(w))\|}{d_\Omega(f(z))},$$

which, together with (2.2) and Theorem A, give that

$$\begin{aligned}
\|D_f(z)\| &= \|D_f(\phi(0))\| = \frac{d_\Omega(f(z))\|D_F(0)\|}{|\phi'(0)|}\\
&\geq \frac{d_\Omega(f(z))\|D_F(0)\|}{1-|z|^2}\\
&\geq \frac{1+K}{2K}\frac{d_\Omega(f(z))}{1-|z|^2}. \qquad (2.3)
\end{aligned}$$

Since Ω is a radial John disk, we can choose $w_0=f(0)$ as the John center and $\gamma=f([0,\rho\zeta])$ as the John curve; Ω can be assumed to be a radial c-John disk with respect to this choice, where $c\geq 1$. Hence for $w=f(r\zeta)$ and $w_1=f(\rho\zeta)$, we have

$$\sigma_\ell(w)\leq c d_\Omega(w) \text{ for all } \rho\in[r,1). \tag{2.4}$$

The boundedness of Ω implies that $d_\Omega(w)$ is finite for all $w \in \Omega$. Hence the limit

$$\lim_{\rho\to 1-}\int_r^\rho |df(t\zeta)| \tag{2.5}$$

does exist and is finite. By (2.4) and (2.5), we get

$$\frac{1}{K}\int_r^1 \|D_f(t\zeta)\|dt \le \int_r^1 l(D_f(t\zeta))dt \le \int_r^1 |df(t\zeta)| \le c d_\Omega(w), \tag{2.6}$$

where $\zeta \in \partial\mathbb{D}$. By (2.3) and (2.6), we have

$$\int_r^1 \|D_f(t\zeta)\|dt \le \frac{2cK^2}{1+K}(1-r^2)\|D_f(r\zeta)\| \le M_0(1-r)\|D_f(r\zeta)\|, \tag{2.7}$$

where $M_0 = \frac{4cK^2}{1+K} \ge 2c$.

Next, we let

$$\varphi(r) = (1-r)^{-\frac{1}{M_0}}\int_r^1 \|D_f(t\zeta)\|dt.$$

By (2.7), we have

$$\varphi'(r) = (1-r)^{-\frac{1}{M_0}}\left[\frac{1}{M_0(1-r)}\int_r^1 \|D_f(t\zeta)\|dt - \|D_f(r\zeta)\|\right] \le 0,$$

which implies that $\varphi(r)$ is decreasing on the unit interval $(0,1)$.

By Theorem B, for $\rho \le t \le \frac{1+\rho}{2}$, there is a positive constant c_1 such that

$$\|D_f(\rho\zeta)\| \le 4^{c_1}\|D_f(t\zeta)\|,$$

which gives

$$\begin{aligned}\int_\rho^1 \|D_f(t\zeta)\|dt &\ge \int_\rho^{\frac{1+\rho}{2}} \|D_f(t\zeta)\|dt\\ &\ge 4^{-c_1}\|D_f(\rho\zeta)\|\int_\rho^{\frac{1+\rho}{2}} dt\\ &= 2^{-2c_1-1}\|D_f(\rho\zeta)\|(1-\rho).\end{aligned} \tag{2.8}$$

For $0 \le r \le \rho < 1$, by (2.7) and (2.8), we have

$$\begin{aligned}(1-\rho)^{1-\frac{1}{M_0}}\|D_f(\rho\zeta)\| &\le 2^{1+2c_1}\varphi(\rho) \le 2^{1+2c_1}\varphi(r)\\ &\le 2^{1+2c_1}M_0(1-r)^{1-\frac{1}{M_0}}\|D_f(r\zeta)\|,\end{aligned}$$

which yields

$$\begin{aligned}\|D_f(\rho\zeta)\| &\le 2^{1+2c_1}M_0\|D_f(r\zeta)\|\left(\frac{1-r}{1-\rho}\right)^{1-\frac{1}{M_0}}\\ &= 2^{1+2c_1}M_0\|D_f(r\zeta)\|\left(\frac{1-\rho}{1-r}\right)^{\frac{1}{M_0}-1}.\end{aligned}$$

The proof of the theorem is complete. □

For $z_1, z_2 \in \mathbb{D}$, the *hyperbolic metric* (or *Poincaré metric*) is defined by

$$\lambda_{\mathbb{D}}(z_1,z_2)=\min_{\gamma}\int_{\gamma}\frac{|dz|}{1-|z|^2},$$

where the minimum is taken over all curves γ in $\mathbb{D}$ from z_1 and z_2. It is well-known that, for $z_1, z_2 \in \mathbb{D}$,

$$\lambda_{\mathbb{D}}(z_1,z_2)=\frac{1}{2}\log\frac{1+|z_1-z_2|/|1-\overline{z}_1z_2|}{1-|z_1-z_2|/|1-\overline{z}_1z_2|},$$

which is equivalent to

$$\left|\frac{z_1-z_2}{1-\overline{z}_1z_2}\right|=\frac{e^{2\lambda_{\mathbb{D}}(z_1,z_2)}-1}{e^{2\lambda_{\mathbb{D}}(z_1,z_2)}+1}=\tanh\lambda_{\mathbb{D}}(z_1,z_2).$$

In [21], Sheil-Small proved the following result.

Lemma C. *Let $f=h+\overline{g}\in\mathcal{S}_H$ and $\alpha=\sup_{f\in\mathcal{S}_H}\frac{|h''(0)|}{2}$, where h and g are analytic in $\mathbb{D}$. Then*

$$\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}\leq|h'(z)|\leq\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}.$$

We remark that $\alpha=\sup_{f\in\mathcal{S}_H}\frac{|h''(0)|}{2}$ is finite, but the sharp upper bound of α is still unknown (see [8, 21]).

Lemma 1. *Let $f=h+\overline{g}\in\mathcal{S}_H$, where h and g are analytic in $\mathbb{D}$. Then, for $z_1,z_2\in\mathbb{D}$,*

$$\frac{1}{2}\|D_f(z_1)\|e^{-(1+\alpha)\lambda_{\mathbb{D}}(z_1,z_2)}\leq\|D_f(z_2)\|\leq2\|D_f(z_1)\|e^{(1+\alpha)\lambda_{\mathbb{D}}(z_1,z_2)},$$

where α is defined in Lemma C.

Proof. Let $f=h+\overline{g}\in\mathcal{S}_H$ and $z=\frac{z_2-z_1}{1-\overline{z}_1z_2}$, where h, g are analytic in $\mathbb{D}$ and $z_1,z_2\in\mathbb{D}$. Then

$$F(z)=\frac{f(z_2)-f(z_1)}{(1-|z_1|^2)h'(z_1)}\in\mathcal{S}_H,$$

where $z_2=\frac{z+z_1}{1+\overline{z}_1z}$. By Lemma C, we get

$$\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}\leq|F_z(z)|=\frac{|h'(z_2)|}{|h'(z_1)||1+\overline{z}_1z|^2}\leq\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}},$$

which gives

$$\frac{(1-|z|)^{\alpha+1}}{(1+|z|)^{\alpha+1}}|h'(z_1)|\leq|h'(z_2)|\leq\frac{(1+|z|)^{\alpha+1}}{(1-|z|)^{\alpha+1}}|h'(z_1)|.\tag{2.9}$$

By (2.9), we obtain

$$\frac{1}{2}\frac{(1-|z|)^{\alpha+1}}{(1+|z|)^{\alpha+1}}\|D_f(z_1)\|\leq\|D_f(z_2)\|\leq2\frac{(1+|z|)^{\alpha+1}}{(1-|z|)^{\alpha+1}}\|D_f(z_1),$$

which implies that

$$\frac{1}{2}\|D_f(z_1)\|e^{-(1+\alpha)\lambda_{\mathbb{D}}(z_1,z_2)}\leq\|D_f(z_2)\|\leq2\|D_f(z_1)\|e^{(1+\alpha)\lambda_{\mathbb{D}}(z_1,z_2)}.$$

The proof of this lemma is complete. □

Lemma 2. *Let a_1, a_2 and a_3 be positive constants and let $0 < |z_0| = 1 - \delta$, where $\delta \in (0, 1)$. If $f \in mathcalS_H$, $0 \le 1 - a_2\delta \le |z| \le 1 - a_1\delta$ and $|\arg z - \arg z_0| \le a_3\delta$, then*

$$\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \le \|D_f(z)\| \le M(a_1, a_2, a_3)\|D_f(z_0)\|,$$

where $M(a_1, a_2, a_3) = 2e^{(1+\alpha)\left(a_3 + \frac{1}{2}\log\frac{2a_2 - a_1}{a_1}\right)}$ and α is defined in Lemma C.

Proof. Let $\angle AOB = 2a_3\delta$ and z_1, z_2, z_3 line in the line OB with $|z_1| \le |z_2| = |z_0| \le |z_3|$, see Figure 1. Then the length of the circular arc from z_0 to z_2 is less than $a_3\delta$.

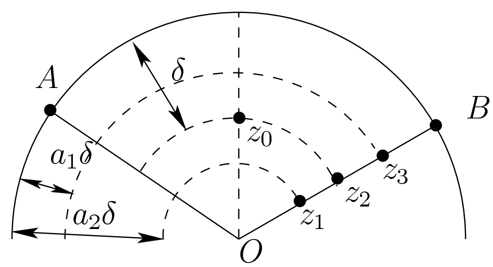

FIGURE 1

By calculations, we have

$$\lambda_{\mathbb{D}}(z_0, z_2) < \frac{a_3\delta}{1 - (1 - \delta)^2} = \frac{a_3}{2 - \delta} < a_3$$

and

$$\left|\frac{z_3 - z_1}{1 - \overline{z}_1 z_3}\right| = \frac{1 - a_1\delta - (1 - a_2\delta)}{1 - (1 - a_1\delta)(1 - a_2\delta)} = \frac{a_2 - a_1}{a_2 + a_1(1 - a_2\delta)} \le \frac{a_2 - a_1}{a_2}.$$

Hence

$$\begin{aligned} \lambda_{\mathbb{D}}(z_0, z) &\le \lambda_{\mathbb{D}}(z_0, z_2) + \lambda_{\mathbb{D}}(z_2, z_1) \\ &\le \lambda_{\mathbb{D}}(z_0, z_2) + \lambda_{\mathbb{D}}(z_1, z_3) \\ &\le a_3 + \frac{1}{2}\log\frac{2a_2 - a_1}{a_1}. \end{aligned}$$

By Lemma 1, we see that

$$\frac{1}{M(a_1, a_2, a_3)} \|D_f(z_0)\| \le \|D_f(z)\| \le M(a_1, a_2, a_3)\|D_f(z_0)\|,$$

where $M(a_1, a_2, a_3)$ is defined as in the statement. □

Proof of Theorem 2. We first prove (c)⇒(b). Let $z = re^{i\theta} \in \mathbb{D}$ and $r_1e^{i\theta_1}, r_2e^{i\theta_2} \in B(re^{i\theta})$ with $r_1 \leq r_2$. Then, by (1.2), Lemma 2 and [16, Proposition 13], there is a positive constant M such that

$$
\begin{aligned}
|f(r_2e^{i\theta_2}) - f(r_1e^{i\theta_1})| &\leq |f(r_2e^{i\theta_2}) - f(re^{i\theta_2})| + |f(r_1e^{i\theta_1}) - f(re^{i\theta_1})| \\
&\quad +|f(re^{i\theta_2}) - f(re^{i\theta_1})| \\
&\leq \int_r^{r_2} \|D_f(\rho e^{i\theta_2})\| d\rho + \int_r^{r_1} \|D_f(\rho e^{i\theta_1})\| d\rho \\
&\quad +r\int_{\gamma_0} \|D_f(re^{it})\| dt \\
&\leq M_2 \int_r^{r_2} \|D_f(re^{i\theta})\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1} d\rho \\
&\quad +M_2 \int_r^{r_1} \|D_f(re^{i\theta})\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1} d\rho \\
&\quad +Mr\int_{\gamma_0} \|D_f(re^{i\theta})\| dt \quad \text{(by Lemma 2)} \\
&\leq \frac{2M_2}{\delta}\|D_f(re^{i\theta})\|(1-r) + Mr\ell(\gamma_0)\|D_f(re^{i\theta})\| \\
&\leq \frac{2M_2}{\delta}\|D_f(re^{i\theta})\|(1-r) + M|\theta_2-\theta_1|\|D_f(re^{i\theta})\| \\
&\leq \left(\frac{2M_2}{\delta} + 2\pi M\right)\|D_f(re^{i\theta})\|(1-r) \\
&\leq 16K\left(\frac{2M_2}{\delta} + 2\pi M\right) d_\Omega(f(z)), \quad \text{by [16, Proposition 13]},
\end{aligned}
$$

where γ_0 is the smaller subarc of $\partial\mathbb{D}_r$ between $re^{i\theta_1}$ and $re^{i\theta_2}$. Hence there exists a positive constant M_1 such that, for all $z \in \mathbb{D}$,

$$\operatorname{diam} f(B(z)) \leq M_1 d_\Omega(f(z)).$$

Next we prove (b)⇒(c). For $z = re^{i\theta} \in \mathbb{D}$, let

$$\phi(r) = \int_r^1 (1-x)\|D_f(xe^{i\theta})\|^2 dx \tag{2.10}$$

and

$$\Delta(r) = \{\zeta = x+iy : \ r \leq x < 1, \ 0 \leq y \leq 1-x\}.$$

By Lemma 2, for $\zeta = x+iy \in \Delta(r)$, there exists a positive constant M_3 such that

$$\|D_f(xe^{i\theta})\| \leq M_3\|D_f(\zeta e^{i\theta})\|,$$

which implies that

$$\begin{aligned}
\phi(r) &\leq \int_r^1 \int_0^{1-x} \|D_f(xe^{i\theta})\|^2 dydx \\
&\leq M_3^2 \int_r^1 \int_0^{1-x} \|D_f(\zeta e^{i\theta})\|^2 dydx \\
&\leq KM_3^2 \int_r^1 \int_0^{1-x} J_f(\zeta e^{i\theta}) dydx \\
&= KM_3^2 A\big(f(\Delta(re^{i\theta}))\big),
\end{aligned} \tag{2.11}$$

where

$$\Delta(re^{i\theta}) = \{\zeta e^{i\theta} = (x+iy)e^{i\theta} :\ r \leq x < 1,\ 0 \leq y \leq 1-x\}.$$

It is not difficult to see that $\Delta(re^{i\theta}) \subset B(re^{i\theta})$, which, together with (2.3) and (2.11), imply

$$\begin{aligned}
\phi(r) &\leq KM_3^2 A\big(f(\Delta(re^{i\theta}))\big) \leq KM_3^2 A\big(f(B(re^{i\theta}))\big) \\
&\leq \frac{\pi KM_3^2}{4} \left(\operatorname{diam}\big(f(B(re^{i\theta}))\big)\right)^2 \\
&\leq \frac{\pi KM_3^2 M_1^2}{4} \left(d_\Omega(f(z))\right)^2 \\
&\leq \frac{\pi K^3 M_1^2 M_3^2}{(1+K)^2} (1-|z|^2)^2 \|D_f(z)\|^2.
\end{aligned} \tag{2.12}$$

By (2.10), for $r \leq \rho < 1$, we get

$$\log\frac{\phi(\rho)}{\phi(r)} = \int_r^\rho \frac{\phi'(t)}{\phi(t)} dt \leq -\alpha \int_r^\rho \frac{dt}{1-t} = \alpha \log\frac{1-\rho}{1-r}, \tag{2.13}$$

where $\alpha = (1+K)^2/\left[\pi K^3 M_1^2 M_3^2\right]$. For $\rho \leq x \leq \frac{1+\rho}{2}$, by Theorem B, there is a positive constant c_1^* such that

$$\|D_f(\rho e^{i\theta})\| \leq 4^{c_1^*} \|D_f(xe^{i\theta})\|. \tag{2.14}$$

Applying (2.10), (2.13) and (2.14), we have

$$\begin{aligned}
\frac{1}{2^{4c_1^*+1}} (1-\rho)^2 \|D_f(\rho e^{i\theta})\|^2 &= \frac{1}{2^{4c_1^*}} \int_\rho^1 (1-x) \|D_f(\rho e^{i\theta})\|^2 dx \\
&\leq \int_\rho^1 (1-x) \|D_f(xe^{i\theta})\|^2 dx = \phi(\rho) \\
&\leq \phi(r) \left(\frac{1-\rho}{1-r}\right)^\alpha,
\end{aligned}$$

which, together with (2.12), yield that

$$\frac{1}{2^{4c_1^*+1}} \|D_f(\rho e^{i\theta})\|^2 (1-\rho)^2 \leq \phi(r) \left(\frac{1-\rho}{1-r}\right)^\alpha \leq \frac{1}{\alpha} (1-r)^2 \|D_f(re^{i\theta})\|^2 \left(\frac{1-\rho}{1-r}\right)^\alpha.$$

Then we conclude that

$$\|D_f(\rho e^{i\theta})\| \leq \sqrt{\frac{2^{1+4c_1^*}}{\alpha}}\|D_f(z)\| \left(\frac{1-\rho}{1-r}\right)^{\frac{\alpha}{2}-1}. \tag{2.15}$$

By (2.15) and Lemma 2, for all $\zeta = \rho e^{i\eta} \in B(z)$, there exists a positive constant M_4 such that

$$\|D_f(\rho e^{i\eta})\| \leq M_4\|D_f(\rho e^{i\theta})\| \leq M_4\sqrt{\frac{2^{1+4c_1^*}}{\alpha}}\|D_f(z)\| \left(\frac{1-\rho}{1-r}\right)^{\frac{\alpha}{2}-1}.$$

Now we prove (a)⇒(c). By Theorem 1, there are constants M and $\delta \in (0,1)$ such that for each $\zeta \in \partial\mathbb{D}$ and for $0 \leq r \leq \rho < 1$,

$$\|D_f(\rho\zeta)\| \leq M\|D_f(r\zeta)\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1}. \tag{2.16}$$

For all $\xi \in \partial\mathbb{D}$ with $|\arg\xi - \arg\zeta| \leq \pi(1-r)$, by Lemma 2, there is a positive constant M' such that

$$\|D_f(r\zeta)\| \leq M'\|D_f(r\xi)\|. \tag{2.17}$$

Hence (1.2) follows from (2.16) and (2.17).

At last, we prove (c)⇒(a). By [16, Proposition 13], (1.2), for $w = f(r\zeta)$ and $w_1 = f(\rho\zeta)$, we have

$$\begin{aligned}
\sigma_\ell(w) &= \int_r^\rho |df(t\zeta)| \leq \int_r^\rho \|D_f(t\zeta)\|dt \\
&\leq M_2\|D_f(r\zeta)\| \int_r^\rho \left(\frac{1-t}{1-r}\right)^{\delta-1} dt = \frac{M_2}{\delta}\|D_f(r\zeta)\|(1-r) \\
&\leq \frac{M_2}{\delta}\|D_f(r\zeta)\|(1-r^2) \\
&\leq \frac{16KM_2}{\delta} d_\Omega(w),
\end{aligned}$$

which implies that Ω is a radial $(16KM_2/\delta)$-John disk with John center 0 and with $\gamma = f([0,\rho\zeta])$ as the John curves, where $r \in [0,1)$, $\rho \in [r,1)$ and $\zeta \in \partial\mathbb{D}$. The proof is complete. □

Proof of Proposition 3. Without loss of generality, we assume that there is a positive constant $M_1^\star$ such that, for all $z \in \mathbb{D}$,

$$\operatorname{diam} f(B(z)) \leq M_1^\star d_\Omega(f(z)), \tag{2.18}$$

where $\Omega = f(\mathbb{D})$. For $r \in [0,1)$, let

$$\begin{aligned}
\varphi(r) &= \frac{1}{2\pi}\int_0^{2\pi} \left(|f_z(re^{it})|^2 + |f_{\overline{z}}(re^{it})|^2\right) dt \\
&= 1 + \sum_{n=2}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2\right) r^{2n-2}.
\end{aligned} \tag{2.19}$$

Then, by Theorem A and (2.18), we obtain

$$
\begin{aligned}
\int_r^1 \int_{-\pi(1-r)}^{\pi(1-r)} J_f(\rho e^{i(\theta+t)})\rho d\theta d\rho &= A\left(f(B(re^{it}))\right) \\
&\leq \frac{\pi}{4}\mathrm{diam}^2\left(f(B(re^{it}))\right) \\
&\leq M^*(1-r^2)^2\|D_f(re^{it})\|^2,
\end{aligned} \tag{2.20}
$$

where $M^* = \frac{\pi K^2 M_1^{\star 2}}{(1+K)^2}$.

By (2.20), for $r \in [\frac{1}{2}, 1)$, we obtain

$$
\begin{aligned}
\frac{1}{2K}\int_r^1 \int_{-\pi(1-r)}^{\pi(1-r)} \varphi(\rho) d\theta d\rho &\leq \frac{1}{K}\int_r^1 \rho\left(\int_0^{2\pi} \|D_f(\rho e^{i(t+\theta)})\|^2 dt\right) d\theta d\rho \\
&\leq \int_r^1 \int_{-\pi(1-r)}^{\pi(1-r)} \rho\left(\int_0^{2\pi} J_f(\rho e^{i(t+\theta)})\right) d\theta d\rho \\
&\leq 4M^*(1-r)^2\int_0^{2\pi} \|D_f(re^{it})\|^2 dt \\
&\leq 16\pi M^*(1-r)^2\varphi(r),
\end{aligned}
$$

which gives that

$$
\int_r^1 \varphi(\rho)d\rho \leq 16KM^*(1-r)\varphi(r) = \beta(1-r)\varphi(r), \tag{2.21}
$$

where $\beta = 16KM^*$. Applying (2.21), for $r \in [\frac{1}{2}, 1)$, we get

$$
\begin{aligned}
&\frac{d}{dr}\left[(1-r)^{-2\beta_0}\int_r^1 \varphi(\rho)d\rho\right] \\
&= \frac{1}{2\beta_0}(1-r)^{-2\beta_0-1}\int_r^1 \varphi(\rho)d\rho - (1-r)^{-2\beta_0}\varphi(r) \leq 0,
\end{aligned} \tag{2.22}
$$

where $\beta_0 = 1/(2\beta)$. By (2.22), for $r \in [\frac{1}{2}, 1)$, we have

$$
(1-r)^{1-2\beta_0}\varphi(r) \leq (1-r)^{-2\beta_0}\int_r^1 \varphi(\rho)d\rho \leq 2^{-2\beta_0}\int_{\frac{1}{2}}^1 \varphi(\rho)d\rho < \infty. \tag{2.23}
$$

It follows from (2.19) and (2.23) that there are two positive constants $M_1^{'}$ and $M_1^{''}$ such that

$$
\begin{aligned}
1 + \sum_{n=2}^{\infty} n^{1+\beta_0}(|a_n|^2 + |b_n|^2) &\leq M_1^{'}\int_{\frac{1}{2}}^1 (1-r)^{-\beta_0}\varphi(r)dr \\
&\leq M_1^{''}\int_{\frac{1}{2}}^1 (1-r)^{\beta_0-1}dr < \infty.
\end{aligned}
$$

The proof of this proposition is complete. $\square$

Lemma 3. *Let $f \in \mathcal{S}_H$ be a K-quasiconformal harmonic mapping from $\mathbb{D}$ onto a bounded domain G. If there are constants M and $\delta \in (0,1)$ such that for each $\varsigma \in \partial\mathbb{D}$ and for $0 \leq r \leq \rho < 1$,*

$$\|D_f(\rho\varsigma)\| \leq M\|D_f(r\varsigma)\| \left(\frac{1-\rho}{1-r}\right)^{\delta-1}, \tag{2.24}$$

then, for $a \in \mathbb{D}$, we have

$$\mathrm{diam} f(I(a)) \leq M_0' d_G(a),$$

where

$$I(a) = \{z \in \partial\mathbb{D} : |\arg z - \arg a| \leq 1 - |a|\}$$

and

$$M_0' = 32K\left(2e^{(1+\alpha)} + \frac{M2e^{(1+\alpha)}}{\delta} + \frac{M}{\delta}\right).$$

Proof. For $a \in \mathbb{D}$, let $a = \rho\zeta$ with $\rho = |a|$. For $z \in I(a)$, by (2.24) and Lemma 2, we have

$$\begin{aligned} |f(z\rho) - f(\rho\zeta)| &\leq \int_{\gamma'} \rho\|D_f(\rho\xi)\|\,|d\xi| \\ &\leq 2e^{(1+\alpha)}\rho\int_{\gamma'} \|D_f(\rho\zeta)\|\,|d\xi|, \quad \text{by Lemma 2,} \\ &= 2e^{(1+\alpha)}\rho\ell(\gamma')\|D_f(\rho\zeta)\| \\ &= 2e^{(1+\alpha)}\rho^2\|D_f(\rho\zeta)\|\,|\arg(\rho\zeta) - \arg z| \\ &\leq 2e^{(1+\alpha)}\rho^2(1-\rho)\|D_f(\rho\zeta)\| \\ &\leq 2e^{(1+\alpha)}(1-\rho)\|D_f(\rho\zeta)\|, \end{aligned} \tag{2.25}$$

$$\begin{aligned} |f(z\rho) - f(z)| &\leq \int_\rho^1 \|D_f(tz)\|\,dt \\ &\leq M\int_\rho^1 \|D_f(\rho z)\|\left(\frac{1-t}{1-\rho}\right)^{\delta-1} dt, \text{ by (2.24),} \\ &= \frac{M}{\delta}(1-\rho)\|D_f(\rho z)\| \\ &\leq \frac{2Me^{(1+\alpha)}}{\delta}(1-\rho)\|D_f(\rho\zeta)\| \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} |f(\zeta\rho) - f(\zeta)| &\leq \int_\rho^1 \|D_f(t\zeta)\| dt \\ &\leq M\int_\rho^1 \|D_f(\rho\zeta)\|\left(\frac{1-t}{1-\rho}\right)^{\delta-1} dt, \text{ by (2.24),} \\ &= \frac{M}{\delta}(1-\rho)\|D_f(\rho\zeta)\|, \end{aligned} \tag{2.27}$$

where γ' is the smaller subarc of $\partial\mathbb{D}_\rho$ between ρz and $\rho\zeta$.

Again, for $z \in I(a)$, by (2.1), (2.25), (2.26) and (2.27), we obtain

$$\begin{aligned} |f(\zeta)-f(z)| &\leq |f(\rho\zeta)-f(\rho z)|+|f(z)-f(\rho z)|+|f(\rho\zeta)-f(\zeta)| \\ &\leq M_1^*(1-\rho)\|D_f(\rho\zeta)\| \\ &\leq 16M_1^*Kd_G(a), \text{ by (2.1)}, \end{aligned}$$

which in turn implies that $\operatorname{diam} f(I(a)) \leq 32KM_1^*d_G(a)$, where

$$M_1^* = 2e^{(1+\alpha)} + \frac{M2e^{(1+\alpha)}}{\delta} + \frac{M}{\delta}. \tag{2.28}$$

The proof of the lemma is complete. $\square$

Proof of Theorem 4. Let $\frac{1}{2} < \nu < 1$ and

$$\sup_{0<r<1}\left\{\sup_{w_1,w_2\in\gamma_r}\frac{\ell\big(\gamma_r[w_1,w_2]\big)}{d_{G_r}(w_1,w_2)}\right\} = M_\gamma, \tag{2.29}$$

where γ_r is given by (1.3). Then, by (2.29), Lemma 3 and [7, Theorem 3], we have

$$\begin{aligned} \frac{\nu}{K}\int_0^{2\pi}\|D_f(\nu e^{i\theta})\|d\theta &\leq \nu\int_0^{2\pi} l\big(D_f(\nu e^{i\theta})\big)d\theta \\ &\leq \int_0^{2\pi}|df(\nu e^{i\theta})| \\ &\leq \sum_{k=1}^{7}\int_{I(z_k)}|df(\nu e^{i\theta})| \\ &\leq M_\gamma\sum_{k=1}^{7}\operatorname{diam} f\left(I(z_k)\right), \text{ by (2.29)}, \\ &\leq 32M_\gamma M_1^*K\sum_{k=1}^{7}d_G\left(f(z_k)\right), \text{ by Lemma 3}, \\ &\leq \frac{64M_\gamma M_1^*K}{1+K}\sum_{k=1}^{7}\left\{(1-|z_k|^2)\|D_f(z_k)\|\right\} \\ &\leq \frac{1792M_\gamma M_1^*K}{(1+K)\pi}, \text{ by [7, Theorem 3]}, \end{aligned}$$

which implies that $\|D_f\| \in H_g^1(\mathbb{D})$, where $k \in \{1,2,...,7\}$,

$$z_k = \frac{1}{2}e^{i(k-1)},\ I(z_k) = \{z \in \partial\mathbb{D} :\ |\arg z - \arg z_k| \leq 1-|z_k|\},$$

and M_1^* is given by (2.28). The proof of the theorem is complete. $\square$

Proof of Theorem 5. By the assumption, we see that there is a $\nu \in (0,1)$ and $r_0 \in (0,1)$ such that, for $r_0 \leq \eta < 1$,

$$\frac{\nu}{1-\eta^2} \geq \mathrm{Re}\big(\zeta P_f(\eta\zeta)\big) = \mathrm{Re}\left(\frac{\zeta h''(\eta\zeta)}{h'(\eta\zeta)}\right) - \mathrm{Re}\left(\frac{\zeta\omega'(\eta\zeta)\overline{\omega(\eta\zeta)}}{1-|\omega(\eta\zeta)|^2}\right),$$

which shows that

$$\begin{aligned}\mathrm{Re}\left(\frac{\zeta h''(\eta\zeta)}{h'(\eta\zeta)}\right) &\leq \mathrm{Re}\left(\frac{\zeta\omega'(\eta\zeta)\overline{\omega(\eta\zeta)}}{1-|\omega(\eta\zeta)|^2}\right) + \frac{\nu}{1-\eta^2} \\ &\leq \frac{|\omega'(\eta\zeta)||\overline{\omega(\eta\zeta)}|}{1-|\omega(\eta\zeta)|^2} + \frac{\nu}{1-\eta^2},\end{aligned} \tag{2.30}$$

where $\zeta \in \partial\mathbb{D}$. By Schwarz-Pick's lemma, we obtain

$$|\omega'(\eta\zeta)| \leq \frac{1-|\omega(\eta\zeta)|^2}{1-\eta^2}. \tag{2.31}$$

By (2.30) and (2.31), we have

$$\mathrm{Re}\left(\frac{\zeta h''(\eta\zeta)}{h'(\eta\zeta)}\right) \leq \frac{1+\nu}{1-\eta^2}.$$

Choosing $\lambda \in (0, 1-\nu)$, there is an $r_1 \in [r_0, 1)$ such that

$$\mathrm{Re}\left(\frac{\zeta h''(\eta\zeta)}{h'(\eta\zeta)}\right) < \frac{2\eta - 2\lambda}{1-\eta^2} \quad \text{for all } \zeta \in \partial\mathbb{D}, \tag{2.32}$$

when $\eta \in [r_1, 1)$. For $0 \leq r_1 \leq r \leq \rho < 1$, by (2.32), we find that

$$\begin{aligned}\log\left[\frac{(1-\rho^2)|h'(\rho\zeta)|}{(1-r^2)|h'(r\zeta)|}\right] &= \int_r^\rho \left[\mathrm{Re}\left(\frac{\zeta h''(\eta\zeta)}{h'(\eta\zeta)}\right) - \frac{2\eta}{1-\eta^2}\right] d\eta \\ &< -2\lambda \int_r^\rho \frac{d\eta}{1-\eta^2} \\ &= -\lambda \log\left(\frac{1+\rho}{1+r}\cdot\frac{1-r}{1-\rho}\right),\end{aligned}$$

which implies that

$$\left|\frac{h'(\rho\zeta)}{h'(r\zeta)}\right| < \left(\frac{1+r}{1+\rho}\right)^{1+\lambda}\left(\frac{1-\rho}{1-r}\right)^{\lambda-1} \leq \left(\frac{1-\rho}{1-r}\right)^{\lambda-1}. \tag{2.33}$$

By (2.33), we get

$$\begin{aligned}\|D_f(\rho\zeta)\| &\leq \frac{2K}{1+K}|h'(\rho\zeta)| < \frac{2K}{1+K}|h'(r\zeta)|\left(\frac{1-\rho}{1-r}\right)^{\lambda-1} \\ &\leq \frac{2K}{1+K}\|D_f(r\zeta)\|\left(\frac{1-\rho}{1-r}\right)^{\lambda-1}.\end{aligned} \tag{2.34}$$

In order to apply Theorem 1 and then to conclude that $\Omega_1 = f(\mathbb{D})$ is a radial John disk, we will use some proof techniques as in the proof of [9, Theorem 3.7] to remove

the restriction $r \geq r_1$ above. For $0 \leq r_1 \leq r \leq \rho < 1$, by (2.1) and (2.34), we see that there is a constant $c(\lambda) > 1$ such that

$$\sigma_\ell(w) \leq c(\lambda) d_{\Omega_1}(w), \tag{2.35}$$

where $w_1 = f(\rho\zeta)$, $w = f(r\zeta)$ and $\gamma = f([0, \rho\zeta])$. It follows from (2.35) that

$$\operatorname{diam}(\gamma[w_1, w]) \leq c(\lambda) d_{\Omega_1}(w). \tag{2.36}$$

Now we consider the case: $0 \leq r \leq r_1 \leq \rho < 1$. Let $\delta_0 = \operatorname{dist}\big(f(\overline{\mathbb{D}}_{r_1}), \partial\Omega_1\big)$ denote the Euclidean distance from $f(\overline{\mathbb{D}}_{r_1})$ to the boundary $\partial\Omega_1$ of Ω_1 and let $\lambda_0 = \operatorname{diam}\big(f(\overline{\mathbb{D}}_{r_1})\big)$. Then

$$\delta_0 > 0 \text{ and } \lambda_0 < \infty. \tag{2.37}$$

For $0 \leq r \leq r_1 \leq \rho < 1$, by the triangle inequality, (2.36) and (2.37), we get

$$\begin{aligned}
\operatorname{diam}(\gamma[w, w_1]) &\leq \operatorname{diam}(\gamma[w, w_0]) + \operatorname{diam}(\gamma[w_0, w_1]) \\
&\leq \lambda_0 + c(\lambda) d_{\Omega_1}(w_0) \\
&\leq \lambda_0 + c(\lambda)(\lambda_0 + \delta_0) \\
&\leq (c(\lambda) + c')\delta_0 \\
&\leq (c(\lambda) + c') d_{\Omega_1}(w),
\end{aligned}$$

where $w_1 = f(\rho\zeta)$, $w = f(r\zeta)$, $w_0 = f(r_1\zeta)$ and $c' = (1 + c(\lambda))\lambda_0/\delta_0$.

The remaining case when $0 \leq r \leq \rho \leq r_1 < 1$ is treated similarly. Therefore, for $0 \leq r \leq \rho < 1$, there is a constant $c_2 > 1$ such that

$$\operatorname{diam}\big(\gamma[w, w_1]\big) \leq c_2 d_{\Omega_1}(w),$$

which implies that $\operatorname{car}_d(\gamma, c_2) \subset \Omega_1$ (cf. [17]), where

$$\operatorname{car}_d(\gamma, c_2) = \bigcup \Big\{ \mathbb{D}\big(w, \operatorname{diam}\big(\gamma[w, w_1]\big)/c_2\big) : \ w \in \gamma \setminus \{f(0), w_1\} \Big\}.$$

It follows from [17, Theorem 2.16] and [17, Part 2.26 in P.17] that Ω_1 is a John disk. For the definition of the diameter of c-carrot, denoted by $\operatorname{car}_d(\gamma, c)$, we refer to [17]. The proof of the theorem is complete. □

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